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CHAPTER ONE

Sraffa's Model for the Joint Production of Commodities by Means of Commodities[1]

CARLO FELICE MANARA

1 Aim of this essay

The present essay aims to analyse the model presented by Piero Sraffa in the second part of his book *Production of Commodities by Means of Commodities* (1960). There already exist analyses of the first part of the book, dealing with single-product industries and circulating capital: for example, the analyses by P. Newman (1962) and V. Dominedò (1962). But to my knowledge there has been no mathematical analysis of the second part of the book, 'Multiple-Product Industries and Fixed Capital'. It is hoped, therefore, that the present analysis may prove useful, and that not solely for the aim, shrewdly identified by Newman (1962), of '*translat*[*ing*] Sraffa's work into the more widely used Walrasian dialect of mathematical economics'. My aim is above all to analyse the logical foundations of Sraffa's treatment and to attempt to enunciate hypotheses that make his model viable. Such hypotheses are not always stated clearly and explicitly by Sraffa, perhaps because he makes very limited use of the mathematical language and so considers it unnecessary to specify the precise conditions under which the relationships of which he writes are capable of having sense.

But one of the advantages of translating arguments expressed in ordinary language into the mathematical 'dialect' is that it enforces a rigorous analysis of assumptions and does not allow one to leave anything to 'intuition' or to 'evidence'. One may risk being led astray if such evidence is uncertain or deceptively convincing. It goes without saying that, since the

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present treatment aims to be mainly mathematical, references to economic content are merely occasional. Where hypotheses are formulated, it is on the explicit understanding that it is left to economists to judge whether they are acceptable or not.

2 Notation

For notational convenience, we shall slightly alter Sraffa's symbols according to certain conventions.

We follow Sraffa in using k to refer to the number (obviously an integer) of commodities and of industries in the economic system under examination.

The formulation of the model we are interested in here is based on the consideration of $2k^2$ quantities of commodities, which can be conveniently indicated as the elements of two square matrices of order k.

We shall call $\mathbf{A}$ and $\mathbf{B}$ these square matrices, their elements being, respectively, a_{ij} and b_{ij} $(i, j = 1, 2, \ldots k)$. In a similar way to Sraffa (1960, section 51), a_{ij} indicates the quantity of the ith commodity that enters as means of production into the jth industry, and b_{ij} indicates the quantity of ith commodity produced by the jth industry. Hence, the rows of matrices $\mathbf{A}$ and $\mathbf{B}$ correspond to the commodities (understood respectively as means of production and as products) and the columns correspond to the industries of the economic system.

We shall indicate by

$$\mathbf{p} = [p_1, p_2, \ldots p_k]$$

the vector the components of which are the prices of individual commodities; therefore the first, second, third, kth component of the vector $\mathbf{p}$ is the price of, respectively, the first, second, third, kth commodity. We shall then indicate by

$$\mathbf{q} = [q_1, q_2, \ldots q_k] \tag{1.1}$$

the vector the components of which are the quantities of labour used by the industries. Finally, we call the rate of profit r (cf. Sraffa, 1960, section 4) and the general wage rate w.

We shall adopt the conventions of matrix algebra as generally used nowadays (cf. Manara–Nicola, 1967, Appendix II): in particular, when we indicate a vector $\mathbf{x}$ we shall consider it every time as a row vector, i. e. as a matrix of the special order $1 \times k$; the column vector having the components of the vector $\mathbf{x}$ will be indicated by the symbol $\mathbf{x}_T$, i. e. as a matrix of order $k \times 1$ obtained from a row vector, i.e. from a matrix of order $1 \times k$, by means of transposition.

In particular, we point out that, having for example indicated a matrix as $\mathbf{A}$ and a vector as $\mathbf{x}$, by the notation

$$\mathbf{A} > \mathbf{0}, \qquad \mathbf{x} > \mathbf{0} \tag{1.2}$$

we mean to indicate that all the elements of the matrix (and the components of the vector) are positive numbers. By the notation

$$\mathbf{A} \geq \mathbf{0} \text{ and respectively } \mathbf{x} \geq \mathbf{0} \tag{1.3}$$

we mean to indicate that all the elements of the matrix (and the components of the vector) are non-negative numbers, and that at least one element (or one component) is a positive number.

Finally, with the notation

$$\mathbf{A} \geqq \mathbf{0} \text{ and respectively } \mathbf{x} \geqq \mathbf{0} \tag{1.4}$$

we mean to indicate that all the elements of the matrix (or, respectively, the components of the vector) are non-negative numbers, not excluding the possibility that they may all be equal to zero.

With the conventions that we have decided to adopt, the fundamental system of equations of Sraffa's model (cf. Sraffa, 1960, Section 51) is written in the single equation

$$\mathbf{pA}(1+r) + w\mathbf{q} = \mathbf{pB} \tag{1.5}$$

Clearly, every component of the vector which is on the left-hand side of (1.5) represents the production cost of a single industry (a cost including the cost of acquiring commodities used as means of production, the reward of capital and wages for labour). The corresponding component of the vector on the right-hand side of (1.5) represents the revenue of the above-mentioned industry.

From the economic meaning of equation (1.5) and of its symbols we can immediately obtain that, for the matrices, the vectors and the constants r and w, the following relationships must be valid:

$$\mathbf{A} \geq \mathbf{0}, \qquad \mathbf{B} \geq \mathbf{0} \tag{1.6}$$

$$\mathbf{p} \geq \mathbf{0}, \qquad \mathbf{q} \geq \mathbf{0} \tag{1.7}$$

$$r \geq 0, \qquad w \geq 0 \tag{1.8}$$

An observation of which we shall later need to make use is that it is possible arbitrarily to reorder the columns of the matrices $\mathbf{A}$ and $\mathbf{B}$, making any interchange, as long as the same interchange is carried out on the components of the vector $\mathbf{q}$. Similarly, it is possible arbitrarily to reorder the rows of matrices $\mathbf{A}$ and $\mathbf{B}$, making any interchange (even an interchange that is different from the interchange that may have been carried out in the columns), as long as the same interchange is carried out on the elements of the vector $\mathbf{p}$.

Since the interchange of the columns of a matrix is obtained by right-hand side multiplication of the matrix by a matrix S, which is the product of suitable exchange matrices (see, for example, Manara–Nicola, 1967, Appendix VI), and since the interchange of the rows of a matrix is obtained by left-hand side multiplication of the matrix by a suitable matrix $\mathbf{Z}^{-1}$, which is the product of suitable exchange matrices, equation (1.5) is equivalent to an analogous equation written in the form

$$\mathbf{p}^*\mathbf{A}^*(1+r)+w\mathbf{q}^* = \mathbf{p}^*\mathbf{B}^* \tag{1.5a}$$

where we have defined:

$$\mathbf{p}^* = \mathbf{p}\mathbf{Z}, \qquad \mathbf{q}^* = \mathbf{q}\mathbf{S}, \quad \mathbf{A}^* = \mathbf{Z}^{-1}\mathbf{A}\mathbf{S} \qquad \mathbf{B}^* = \mathbf{Z}^{-1}\mathbf{B}\mathbf{S} \tag{1.9}$$

It is worthwhile pointing out explicitly that the situation we are referring to here does not occur in the case of single-product industries, considered in the first part of Sraffa's book. Indeed, in the matrix considered there, it is not possible to reorder the rows and columns with interchanges which are different from one another, owing to the different meaning that the elements of the matrices have in that case.

3 Conditions of viability of the price system

We now propose to examine the conditions under which the fundamental equation (1.5) – the equation which, as we have pointed out, establishes the balance between revenues and expenditures of the various industries of the economic system under consideration – is viable. In fact, it appears from Sraffa's analysis that the purpose of the vector equation (1.5) (or of the system of equivalent equations given in Sraffa, 1960, section 51) is to determine the commodity prices when the other elements of the equation are fixed. It is quite obvious that such prices must constitute the components of a positive vector, i.e. of a vector satisfying (1.7). Let us suppose for the sake of simplicity that all the commodities under consideration are basic commodities (we shall come back later to the distinction between basic and non-basic commodities). The simplifying hypothesis that we propose here yields the consequence that equation (1.5) should be sufficient to determine the price vector – of course, when certain conditions are fulfilled. These conditions are not enunciated in Sraffa's work, and it is upon them that we shall dwell at this point.

With this in mind, let us write the fundamental equation (1.5) in the following form:

$$w\mathbf{q} = \mathbf{p}[\mathbf{B}-\mathbf{A}(1+r)] \tag{1.10}$$

The analysis that we intend to make has the aim of investigating the conditions that must be fulfilled by matrix $[\mathbf{B}-\mathbf{A}]$, or, more generally, by matrix $[\mathbf{B}-\mathbf{A}(1+r)]$ with

$$r \geq 0 \tag{1.11}$$

so that the vector equation (1.10) is solvable and gives a positive price vector as a result.

The necessity of precisely indicating the conditions under which this situation can come about leads us to the enunciation of certain basic hypotheses (which we shall label UA – 'unstated assumption' – 1, 2, etc.) chosen from among the many possible. As concerns their economic significance, it has already been mentioned that we intend to accept the opinion of economists, who are better able to evaluate the soundness of the hypotheses themselves and of their economic implications. We shall limit ourselves to pointing out that without these hypotheses (or equivalents) the model represented by (1.10) would not be 'viable'.

UA 1. The overall quantity of every commodity used as a means of production is less than the total quantity of the same commodity produced in the whole economic system.

In the vector notation we are adopting, defining

$$\mathbf{s} = [1, 1, \ldots 1] \tag{1.12}$$

hypothesis UA 1 may be expressed by the following relation:

$$[\mathbf{B}-\mathbf{A}]\mathbf{s}_T > \mathbf{0}_T \tag{1.13}$$

UA 2. There exists at least one positive vector of prices, $\hat{\mathbf{p}}$, such that the value of the commodities used as means of production by every individual industry, evaluated at those prices, is smaller than the value of the products, also evaluated at those same prices.

This hypothesis may be translated into the following formula:

$$\exists \hat{\mathbf{p}} \{\hat{\mathbf{p}} > \mathbf{0} \wedge \hat{\mathbf{p}}[\mathbf{B}-\mathbf{A}] > \mathbf{0}\} \tag{1.14}$$

This hypothesis is analogous to that implicitly offered by Leontief (1951) for the 'viability' of his model.

We shall now indicate with X the set of column vectors with non-negative components, i.e. define

$$X = \{\mathbf{x}_T | \mathbf{x}_T \geqq \mathbf{0}_T\} \tag{1.15}$$

Let us then denote with $U(r)$ the set of column vectors belonging to X and such that, for every vector $\mathbf{x}_T$ of $U(r)$, the following relation holds:

$$[\mathbf{B}-\mathbf{A}(1+r)]\mathbf{x}_T \geq \mathbf{0}_T \tag{1.16}$$

In other words, we define

$$U(r) = \{\mathbf{x}_T | \mathbf{x}_T \in X \wedge [\mathbf{B} - \mathbf{A}(1+r)]\mathbf{x}_T \geq \mathbf{0}_T\} \tag{1.17}$$

It is easy to prove that the set $U(r)$ is a convex polyhedral cone. Since clearly

$$\mathbf{s} \in X \tag{1.18}$$

it follows immediately from hypothesis (1.14) that there exists a value of r (to be exact, the value $r = 0$) at which the set $U(r)$ is not empty.

Since the vector function of the real variable r given by the expression

$$[\mathbf{B} - \mathbf{A}(1+r)]\mathbf{x}_T \tag{1.19}$$

is clearly continuous, we can easily deduce from what has been said so far that the set of values of r (belonging to the half-line defined by relation (1.11) in correspondence to which the set $U(r)$ is not empty) is an interval closed on the left-hand side and not empty. Similarly, let us denote with P the set of vectors having non-negative components; i.e. let us define

$$P = \{\mathbf{y} | \mathbf{y} \geqq 0\} \tag{1.20}$$

We shall call $V(r)$ the set of non-negative vectors such that, for every vector $\mathbf{y}$ of $V(r)$, the following relation holds:

$$\mathbf{y}[\mathbf{B} - \mathbf{A}(1+r)] \geq \mathbf{0} \tag{1.21}$$

In other words, we define

$$V(r) = \{\mathbf{y} | \mathbf{y} \in P \wedge \mathbf{y}[\mathbf{B} - \mathbf{A}(1+r)] \geq \mathbf{0}\} \tag{1.22}$$

It is easy to demonstrate that set $V(r)$ is also a convex polyhedral cone.

From hypothesis (1.14) it follows that

$$\hat{\mathbf{p}} \in P \tag{1.23}$$

Consequently, it follows from hypothesis (1.13) that, at least for one value of r (to be exact, for $r = 0$), the set $V(r)$ is not empty. Since the vector function of the real variable r given by the expression

$$\mathbf{y}[\mathbf{B} - \mathbf{A}(1+r)] \tag{1.24}$$

is clearly continuous, it then follows, from what has been said so far, that the set of values of r (belonging to the half-line defined by the relation (1.11) in correspondence to which the set $V(r)$ is not empty) is an interval closed on the left-hand side.

Let us now make the following hypothesis:

UA 3. $$\det[\mathbf{B} - \mathbf{A}] \neq 0 \tag{1.25}$$

This ensures that, for at least one value of r (the value $r = 0$), the vectors forming the rows of the matrix $[\mathbf{B} - \mathbf{A}(1+r)]$ are linearly independent.

Since the real function $f(r)$ of the variable r defined by

$$f(r) = \det[\mathbf{B} - \mathbf{A}(1+r)] \tag{1.26}$$

is clearly continuous, the set of values of r belonging to the half-line defined by relation (1.11) and such that

$$\det[\mathbf{B} - \mathbf{A}(1+r)] \neq 0 \tag{1.27}$$

is an interval closed on the left-hand side, and having $r = 0$ as its minimum.

Let us consider the values of r belonging to the half-line defined by relation (1.11) for which both sets $U(r)$ and $V(r)$ are not empty and for which (1.27) holds. From now on we shall, for convenience, use $\mathscr{I}$ to refer to this interval.

4 A further condition for the viability of the price system

Hypotheses UA 1, UA 2 and UA 3, explicitly stated in section 3, are necessary if the model under consideration is to have solutions with economic meaning. However, they are not yet sufficient. Indeed, if we consider the fundamental equation of the model, which for the convenience of the reader we shall write again in the form of (1.10),

$$w\mathbf{q} = \mathbf{p}[\mathbf{B} - \mathbf{A}(1+r)]$$

it is evident that the equation itself does not possess as a solution a price vector which is positive for *any* vector $\mathbf{q}$ of the quantity of labour absorbed by the industries of the system.

The validity of this is proved by the following example. Assume that

$$k = 3 \tag{1.28}$$

and consider the matrices $\mathbf{A}$ and $\mathbf{B}$ given in the following manner:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix} \tag{1.29}$$

$$\mathbf{B} = \begin{bmatrix} 2{\cdot}9 & 1{\cdot}2 & 1{\cdot}9 \\ 1{\cdot}2 & 2{\cdot}9 & 3{\cdot}9 \\ 0{\cdot}1 & 1{\cdot}2 & 3{\cdot}9 \end{bmatrix} \tag{1.30}$$

These two matrices satisfy hypotheses UA 1 and UA 2, the latter being clearly satisfied when we assume a vector $\hat{\mathbf{p}}$ given by

$$\hat{\mathbf{p}} = [1, 1, 1] \tag{1.31}$$

It is easy to verify that hypothesis UA 3 is also satisfied.

It is also possible to verify that all the commodities considered in the model under examination are basic commodities: the reader may check this (we ask him to accept it for the time being) after examining our treatment, in section 6, of the problem of basic commodities. He will then, by the criteria advanced there, also be able to judge whether a particular model, with matrices such as **A** and **B**, also admits the existence of commodities that are non-basic, in accordance with the terminology and definition given by Sraffa (1960, section 58).

On the other hand, we can verify that with

$$r = 0, \qquad w = 1 \tag{1.32}$$

the vector of the quantities of labour given by

$$\mathbf{q} = [1{\cdot}73,\ 1{\cdot}66,\ 0{\cdot}47] \tag{1.33}$$

yields the following price vector:

$$\mathbf{p} = [11,\ 1,\ -0{\cdot}7] \tag{1.34}$$

– that is, a vector of prices which are not all positive.

From the example just examined we can infer that, for the model to be 'viable', we must state some further hypothesis which will provide the conditions under which, in equation (1.10), given the values of r and w, a vector $\mathbf{q}$ of quantities of labour yields a positive vector of prices, at least under the restrictive hypothesis we have accepted, i.e. the hypothesis that all commodities under consideration are basic commodities.

In order to state such a hypothesis, consider a matrix $[\mathbf{B} - \mathbf{A}(1 + r)]$, corresponding to a value of r belonging to the interval defined at the end of section 3. Let us call $V'(r)$ the set of vectors $\mathbf{z}$ given by the formula

$$\mathbf{z} = \mathbf{p}[\mathbf{B} - \mathbf{A}(1 + r)] \tag{1.35}$$

when $\mathbf{p}$ belongs to the set $V(r)$. The set $V'(r)$ could be called the 'image' of $V(r)$ by the linear application given by the square matrix $[\mathbf{B} - \mathbf{A}(1 + r)]$; it is defined by the formula

$$V'(r) = \{\mathbf{z} \mid \mathbf{z} = \mathbf{p}[\mathbf{B} - \mathbf{A}(1 + r)] \wedge \mathbf{p} \in V(r)\} \tag{1.36}$$

From the definition that we have given of $V'(r)$ it follows immediately that

$$\mathbf{q} \in V'(r) \rightarrow \mathbf{p} = \mathbf{q}[\mathbf{A} - \mathbf{B}(1 + r)]^{-1} > \mathbf{0} \tag{1.37}$$

Hence we may state the hypothesis we have in mind in the following way.

UA 4. For any given value of r belonging to the interval $\mathscr{I}$, the vector $\mathbf{q}$ belongs to the set $V'(r)$. In mathematical terms:

$$r \in \mathscr{I} \rightarrow \mathbf{q} \in V'(r) \tag{1.38}$$

5 A case of non-existence of the 'standard system'

The importance of the standard product in Sraffa's system is well known: in the case of single-product industries (dealt with in the first half of his book) it serves as standard for measuring the value of overall product and for measuring the wage rate and prices. The standard product seems to have a similar importance in the case of joint production. In this second case, however, it seems that also Sraffa realises the potential complications in the definition of the standard product. At least we can interpret in this sense Sraffa's assertion (Sraffa, 1960, p. 47) that for the construction of the standard product negative multipliers must also be considered.

Nevertheless, it does not appear that Sraffa has experienced the slightest doubt concerning the possibility of imagining the existence of a standard product, even though this possibility is not generally verified, but must be postulated by means of a suitable hypothesis on the matrices which we have called **A** and **B**.

Further to clarify this statement, we point out that the set of multipliers which give rise to the standard product is defined by the equation

$$(1+R)\mathbf{A}\,\mathbf{x}_T = \mathbf{B}\,\mathbf{x}_T \tag{1.39}$$

This vector equation is obtained from (1.5) by setting

$$r = R, \qquad w = 0 \tag{1.40}$$

The components of the vector $\mathbf{x}_T$ (defined but for a common multiplicative factor) are coefficients of the linear combination of industries which give rise to the standard product. The equation (1.39) is a translation of the system given by Sraffa (1960, section 63). Here we assume that all the commodities considered are basic commodities. It will be explained why this hypothesis is not restrictive, and the reader will be able to translate our equation, after the existence of non-basic products has been discussed (section 6 below).

Equation (1.39) may also be written in the following form:

$$[\mathbf{B} - \mathbf{A}(1+R)]\,\mathbf{x}_T = \mathbf{0}_T \tag{1.41}$$

In accordance with classical theorems of algebra for systems of linear equations, this equation may be satisfied by a vector $\mathbf{x}_T$, other than the null vector, only if

$$\det[\mathbf{B} - \mathbf{A}(1+R)] = 0 \tag{1.42}$$

In addition, the economic meaning that Sraffa attributes to the standard product makes sense if, and only if, the vector $\mathbf{x}_T$, associated with a value R that satisfies (1.42), is defined as unique but for a multiplicative factor. This occurs if the value

$$r = R \tag{1.43}$$

is a simple root of the algebraic equation

$$\det[\mathbf{B} - \mathbf{A}(1+r)] = 0 \tag{1.44}$$

All these conditions are verified for Sraffa's model in the case of single-product industries, on the basis of well known theorems of Perron and Frobenius. But, in the case of the model that we are interested in here, these conditions may not be fulfilled. This is shown in the following example, where it is not even possible to construct the standard product – at least, if we remain in the field of real numbers.

Consider a model in which

$$k = 2 \tag{1.45}$$

and in which we have

$$\mathbf{A} = \begin{bmatrix} 1 & 1{\cdot}1 \\ 1{\cdot}1 & 1 \end{bmatrix} \tag{1.46}$$

$$\mathbf{B} = \begin{bmatrix} 1{\cdot}09 & 1{\cdot}144 \\ 1{\cdot}144 & 0{\cdot}99 \end{bmatrix} \tag{1.47}$$

If we define

$$1 + r = t \tag{1.48}$$

then we can easily see that the equation

$$\det[\mathbf{B} - \mathbf{A}\,t] = 0 \tag{1.49}$$

becomes, in this case,

$$0{\cdot}21\,t^2 - 0{\cdot}4368\,t + 0{\cdot}229636 = 0 \tag{1.50}$$

which does not have any real root.[2]

On the other hand, it is easy to verify that the matrices **A** and **B**, given respectively by (1.46) and (1.47), satisfy hypotheses UA 1, UA 2 and UA 3.

Consequently, for Sraffa's propositions to hold, it is necessary to add the following hypothesis.

UA 5. The algebraic equation, in the unknown r,

$$\det[\mathbf{B} - \mathbf{A}(1+r)] = 0 \tag{1.51}$$

has at least one real and positive root. This root (or the smallest of the real

[2] The terms 'real number' and 'real solution' are used here in the precise technical sense of mathematics and not in the rather vague sense adopted by Sraffa (1960, section 50). In this section, indeed, so far as I can understand, Sraffa uses the expression 'real solutions' to mean, perhaps, 'solutions that have economic meaning' or 'solutions that have a correspondence in reality'.

and positive roots, if there is more than one) is a simple root of the algebraic equation (1.51).

The final clause, which postulates that the root, or the smallest root, must be a simple root of equation (1.51), is based on the following considerations. From what is said by Sraffa (1960, section 64) it appears that, for reasons inherent to the economic meaning of the standard product, he wishes to adopt the convention that, if there is more than one positive root, then the smallest one, which we may call ρ, is to be taken as the root of equation (1.51) for the construction of the standard product. However, if the construction of the standard product is to have sense, it is necessary that the corresponding equation (1.41) have only one solution vector, defined but for a multiplicative constant. Indeed, the circumstance that would result if (1.41) had at least two linearly independent vectors as solutions would be contrary to Sraffa's intentions. But this could only occur if the root ρ is not a simple root for equation (1.51).

6 The distinction between basic and non-basic commodities

As is well known, the distinction between basic and non-basic commodities is essential to Sraffa's analysis; this is because, among other reasons, according to his point of view, it is the former that determine the vector of prices that satisfies the fundamental equation of his model.

The distinction between basic and non-basic commodities is given in Sraffa (1960, sections 58 ff.) and will be translated into the mathematical notation adopted here. To this end, let us suppose that certain m commodities of our economic system are non-basic. Clearly,

$$m < k \tag{1.52}$$

and for convenience we may assume that

$$k = j + m \qquad (j > 0) \tag{1.53}$$

Making use of the remark stated in section 2, we can imagine that we have reordered the rows of matrices $\mathbf{A}$ and $\mathbf{B}$ (and consequently also the elements of the vector $\mathbf{p}$) so that the commodities we are interested in correspond to the last m rows of the matrices themselves.

To make things clearer, after this reordering, we can consider each of the matrices $\mathbf{A}$ and $\mathbf{B}$ as partitioned into two other matrices: we shall call these $\mathbf{A}'$, $\mathbf{A}''$ and $\mathbf{B}'$, $\mathbf{B}''$ respectively. $\mathbf{A}'$ and $\mathbf{B}'$ are rectangular matrices of order $j \times k$, while matrices $\mathbf{A}''$ and $\mathbf{B}''$ are also rectangular but of order $m \times k$. We therefore write

$$\mathbf{A} = \left[\frac{\mathbf{A}'}{\mathbf{A}''}\right], \qquad \mathbf{B} = \left[\frac{\mathbf{B}'}{\mathbf{B}''}\right] \tag{1.54}$$

By means of matrices $\mathbf{A}''$ and $\mathbf{B}''$ we now construct a $2m \times k$ matrix $\mathbf{D}$, as follows:

$$\mathbf{D} = \left[\frac{\mathbf{A}''}{\mathbf{B}''}\right] \tag{1.55}$$

According to the ideas put forward by Sraffa, if the m commodities corresponding to the rows forming matrices $\mathbf{A}''$ and $\mathbf{B}''$ are non-basic, then matrix $\mathbf{D}$ is of rank m.

In other words, of the columns of matrix $\mathbf{D}$, only m columns are linearly independent, and therefore all the others can be obtained by a linear combination of these.

Making use once more of the remark stated in section 2, we can think in terms of having reordered the columns of matrix $\mathbf{A}$ and matrix $\mathbf{B}$ (and therefore also the elements of vector $\mathbf{q}$), in such a way that the m columns that form a base for the columns of matrix $\mathbf{D}$ are the last columns of such matrices.

Assuming that this reordering has been carried out, the condition stated by Sraffa for the m commodities corresponding to the last m rows of matrices $\mathbf{A}$ and $\mathbf{B}$ to be non-basic can be translated in the following way.

Let us imagine that each of the matrices $\mathbf{A}$ and $\mathbf{B}$ is partitioned into four sub-matrices, $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$ and $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{B}_{21}$, $\mathbf{B}_{22}$, respectively. Matrices $\mathbf{A}_{11}$, $\mathbf{B}_{11}$ are square of order j; matrices $\mathbf{A}_{22}$, $\mathbf{B}_{22}$ are also square of order m. We have, therefore,

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right], \qquad \mathbf{B} = \left[\begin{array}{c|c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right] \tag{1.56}$$

The commodities corresponding to the last m rows of the two matrices $\mathbf{A}$ and $\mathbf{B}$ are non-basic, according to Sraffa's definition, if there exists a matrix $\mathbf{T}$ of order $m \times j$ such that

$$\mathbf{A}_{21} = \mathbf{A}_{22}\mathbf{T}, \qquad \mathbf{B}_{21} = \mathbf{B}_{22}\mathbf{T} \tag{1.57}$$

Matrix $\mathbf{T}$ is a matrix obtained from the coefficients of the linear combination by means of which the first j columns of matrices $\mathbf{A}''$ and $\mathbf{B}''$ (obviously after the reordering we have referred to) are expressed by means of the last m columns.

We may therefore construct the matrix $\mathbf{M}$, square and of order k, in the following way:

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{I}_j & \mathbf{0} \\ \hline -\mathbf{T} & \mathbf{I}_m \end{array}\right] \tag{1.58}$$

where $\mathbf{I}_j$ and $\mathbf{I}_m$ are the identity matrices, of orders j and m respectively.

On the basis of (1.57) we easily see that the matrices

$$\overline{\mathbf{A}} = \mathbf{A}\,\mathbf{M}, \qquad \overline{\mathbf{B}} = \mathbf{B}\,\mathbf{M} \tag{1.59}$$

have the following form:

$$\overline{\mathbf{A}} = \left[\begin{array}{c|c} \overline{\mathbf{A}}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right], \qquad \overline{\mathbf{B}} = \left[\begin{array}{c|c} \overline{\mathbf{B}}_{11} & \mathbf{B}_{12} \\ \hline \mathbf{0} & \mathbf{B}_{22} \end{array}\right] \tag{1.60}$$

where, in particular,

$$\overline{\mathbf{A}}_{11} = \mathbf{A}_{11} - \mathbf{A}_{12}\,\mathbf{T}, \qquad \overline{\mathbf{B}}_{11} = \mathbf{B}_{11} - \mathbf{B}_{12}\,\mathbf{T} \tag{1.61}$$

We may now separate in vector $\mathbf{q}$ the first j components from the last m. We may write, therefore,

$$\mathbf{q} = [\mathbf{q}^1 | \mathbf{q}^2] \tag{1.62}$$

where, as already said, $\mathbf{q}^1$ has j components and $\mathbf{q}^2$ has m components.

Let us now put

$$\bar{\mathbf{q}} = \mathbf{q}\,\mathbf{M} \tag{1.63}$$

It then follows that

$$\bar{\mathbf{q}} = [\bar{\mathbf{q}}^1 | \mathbf{q}^2], \tag{1.64}$$

where clearly

$$\bar{\mathbf{q}} = \mathbf{q}^1 - \mathbf{q}^2\,\mathbf{T} \tag{1.65}$$

Similarly, we may consider the vector of prices $\mathbf{p}$ as partitioned into two sub-vectors of j and m components respectively, and we may write, on analogy with (1.62),

$$\mathbf{p} = [\mathbf{p}^1 | \mathbf{p}^2] \tag{1.66}$$

We may finally imagine that we have multiplied both sides of the fundamental equation of the model

$$\mathbf{p}\,\mathbf{A}(1+r) + w\,\mathbf{q} = \mathbf{p}\,\mathbf{B} \tag{1.67}$$

on the right by matrix **M**.

Having done this, and after multiplication on the right by matrix **M**, equation (1.67) may be replaced by the system of the following two equations:

$$\begin{cases} \mathbf{p}^1\,\overline{\mathbf{A}}_{11}(1+r) + w\,\bar{\mathbf{q}}^1 = \mathbf{p}^1\,\overline{\mathbf{B}}_{11} & (1.68) \\ \mathbf{p}^1\,\mathbf{A}_{12}(1+r) + \mathbf{p}^2\,\mathbf{A}_{22}(1+r) + w\,\mathbf{q}^2 = \mathbf{p}^1\,\overline{\mathbf{B}}_{11} + \mathbf{p}^2\,\mathbf{B}_{22} & (1.69) \end{cases}$$

The vector equation (1.69) is a translation of the system of equations given by Sraffa (1960, section 62). We should point out, however, that this system is not to be considered equivalent to the system which translates the fundamental vector equation (1.67) – at least, not if we wish to preserve for the term 'equivalent' the meaning that it has in the theory of the systems of linear equations. We need only point out that the system which translates equation (1.68) has a different number of equations and unknowns (to be exact, fewer) from the system that translates equation (1.69).

Strictly speaking, only the system of the pair of equations (1.68) and (1.69) is equivalent to equation (1.67), in the sense that every solution of the pair of equations (1.68) and (1.69) supplies a solution to equation (1.67) and vice versa. However, it should be pointed out that the pair of equations (1.68) and (1.69) can be solved in the order in which they have been written; indeed, equation (1.68) involves only vector $\mathbf{p}^1$, the only components of which are the prices of the basic commodities. Once such prices have been determined, it is also possible to solve equation (1.69), determining the prices of the other commodities – when, of course, the conditions allowing such solutions are satisfied. All remarks that we have made concerning the fundamental equation (1.67), when all the products are basic products, may now be made about equation (1.68). Indeed, not any choice of vector $\bar{\mathbf{q}}^1$ leads to a solution that includes all positive components of the price vector $\mathbf{p}^1$. On this particular point we should have to state hypotheses similar to UA 4 (see end of section 4). However, it may finally be pointed out that, in the case of the vector equation (1.68), the vector $\bar{\mathbf{q}}^1$ which appears there and is given by (1.65) may not have positive components. Now, for the purpose of constructing a standard product and in order to highlight the fact that some commodities may not be basic, Sraffa gives an interpretation of the fact that a linear combination of industries may yield coefficients that are not all positive (see the development of this argument in Sraffa, 1960, section 56). But it seems that he has not thought it necessary to interpret negative quantities of labour absorbed by industries. However, such a case needs to be justified or interpreted, and we willingly leave this task to the economists.

The construction of the standard product in the case of equation (1.68) must be carried out following the procedure examined in section 5. Consequently the possibility of such a construction must be ensured by a hypothesis similar to UA 5, since it is not ensured by hypotheses UA 1, UA 2, UA 3 and UA 4 alone.

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